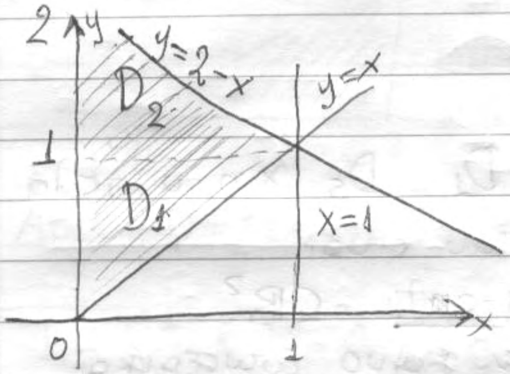
$\int_0^1 \int_x^{2-x} \frac{x}{y}\, dy\, dx \quad D = \{(x,y) : 0 \le x \le 1,\ x \le y \le 2-x\}$ x-απλό



$D_1 = \{(x,y) : 0 \le y \le 1,\ 0 \le x \le y\}$

$D_2 = \{(x,y) : 1 \le y \le 2,\ 0 \le x \le 2-y\}$

$I = -1 + \ln 4$

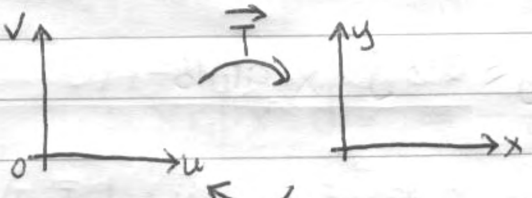
- Παράδειγμα 10.17

15/05/06

Αλλαγή μεταβλητών διπλού ολοκληρώματος

$$\left(\left. \begin{array}{l} \int_a^b f(x)\,dx \\ x = \phi(t),\ t \in [c,d] \\ dx = \phi'(t)\,dt \end{array} \right\} = \int_c^d f(\phi(t))\,\phi'(t)\,dt \right)$$

$G \subseteq \mathbb{R}^2$ υ-απλό (ή v-απλό)



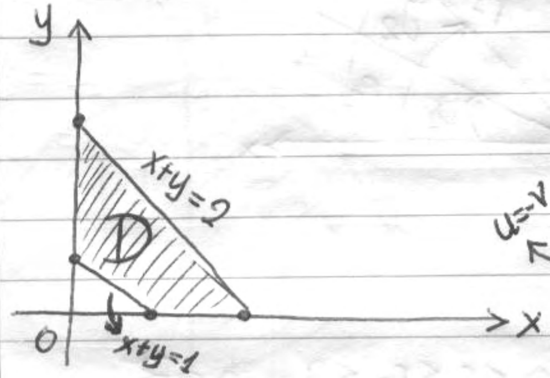
$\vec{T} = \vec{T}(u,v) = (x(u,v),\ y(u,v)) : G \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, C^1

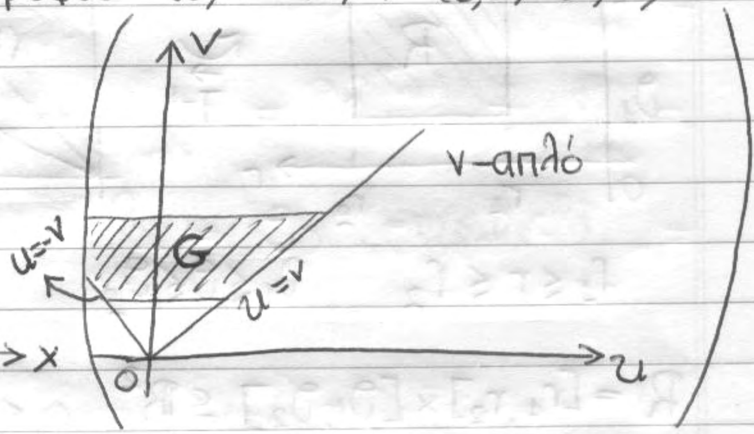
$f : D = \vec{T}(G) \subseteq \mathbb{R}^2 \to \mathbb{R}$ ολοκληρώσιμη

$\Rightarrow$ $$\boxed{\begin{array}{l} \iint_D f(x,y)\,dx\,dy = \iint_G f(x(u,v),\ y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv \\ A(D) = \iint_G \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv \end{array}}$$

• $\iint_D e^{\frac{y-x}{x+y}}\,dx\,dy = I$

D τραπεζοειδές με κορυφές $(0,1)$ $(0,2)$ $(2,0)$ $(1,0)$





θέτω $u = y - x \quad v = x + y$

$\Rightarrow x = \frac{1}{2}(v-u) \quad y = \frac{1}{2}(u+v)$

$x = 0 \rightarrow u = v$

$x+y=2 \rightarrow v = 2$

$y = 0 \rightarrow u = -v$

$x+y=1 \rightarrow v = 1$

$D = \vec{T}(G)$

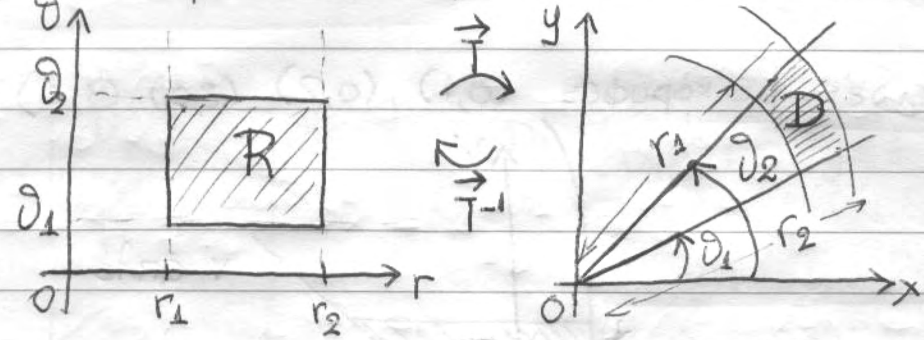
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$$I = \iint_G e^{\frac{u}{v}} \left|-\frac{1}{2}\right| du\,dv = \int_1^2 \int_{-v}^{v} e^{\frac{u}{v}} \frac{1}{2}\,du\,dv =$$

$$= \frac{1}{2}\int_1^2 v \cdot e^{\frac{u}{v}} \Big|_{u=-v}^{u=v} dv = \frac{1}{2}\int_1^2 (e - e^{-1})\,v\,dv =$$

$$= \frac{1}{2}(e - e^{-1}) \int_1^2 v\,dv = \frac{3}{4}(e - e^{-1})$$

• Πολικός μετασχηματισμός



$r_1 \leq r \leq r_2 \qquad \vartheta_1 \leq \vartheta \leq \vartheta_2$

$R = [r_1, r_2] \times [\vartheta_1, \vartheta_2] \subseteq \mathbb{R}^2 \quad 0 \leq r_1 \leq r_2 \leq a \;, a \in \mathbb{R}$

και $\; 0 \leq \vartheta_1 \leq \vartheta_2 \leq 2\pi$

$x = x(r,\vartheta)$
$y = y(r,\vartheta)$ $\Rightarrow$ $\underline{x = r\,\sigma\upsilon\nu\vartheta}$
$\underline{y = r\,\eta\mu\vartheta}$

$$\frac{\partial(x,y)}{\partial(r,\vartheta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \vartheta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \vartheta} \end{vmatrix} = \begin{vmatrix} \sigma\upsilon\nu\vartheta & -r\,\eta\mu\vartheta \\ \eta\mu\vartheta & r\,\sigma\upsilon\nu\vartheta \end{vmatrix} = \underline{r}$$

$f : D = \vec{T}(R) \subseteq \mathbb{R}^2 \to \mathbb{R}$, ολοκλ.

$$\underline{\iint_D f(x,y)\,dx\,dy = \iint_R f(r\,\sigma\upsilon\nu\vartheta, r\,\eta\mu\vartheta)\cdot r \cdot dr\,d\vartheta} =$$

$$\underline{= \int_{\vartheta_1}^{\vartheta_2} \int_{r_1}^{r_2} f(r\,\sigma\upsilon\nu\vartheta, r\,\eta\mu\vartheta)\,dr\,d\vartheta}$$

$$\underline{A(D) = \iint_R r\,dr\,d\vartheta}$$

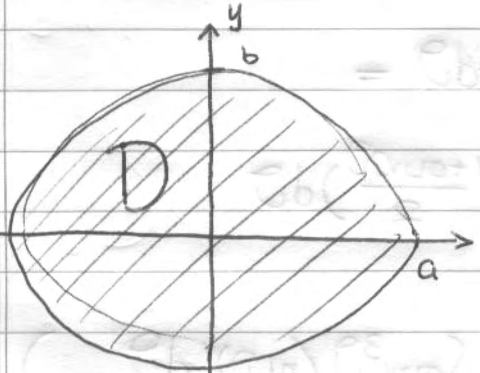
• $G = \{ (r,\vartheta) : \vartheta_1 \leq \vartheta \leq \vartheta_2 , \; \underline{r_1(\vartheta) \leq r \leq r_2(\vartheta)} \}$ ϑ-απλό

$$\iint_D f(x,y)\,dx\,dy = \underline{\int_{\vartheta_1}^{\vartheta_2} \int_{r_1(\vartheta)}^{r_2(\vartheta)} f(r\,\sigma\upsilon\nu\vartheta, r\,\eta\mu\vartheta)\, r\,dr\,d\vartheta}$$

$$\underline{A(D) = \int_{\vartheta_1}^{\vartheta_2} \int_{r_1(\vartheta)}^{r_2(\vartheta)} r\,dr\,d\vartheta}$$

• $I=\iint_D \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{3/2} dxdy$

$D=\{(x,y)\in\mathbb{R}^2 : \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>0,\ b>0\}$



$\underline{x = a r \sigma\upsilon\nu\vartheta}$ $\quad 0\leq r\leq 1$

$\underline{y = b r \eta\mu\vartheta}$ $\quad 0\leq\vartheta\leq 2\pi$

$I=\int_0^{2\pi}\int_0^1 (1-r^2)^{3/2}\, abr\,dr\,d\vartheta =$

$= ab\int_0^{2\pi}\int_0^1 (1-r^2)^{3/2}\, r\,dr\cdot d\vartheta =$

$= -\frac{1}{2}ab\int_0^{2\pi} \frac{(1-r^2)^{3/2}}{\frac{3}{2}+1}\Big|_0^1 d\vartheta = \frac{2}{5}ab\pi$

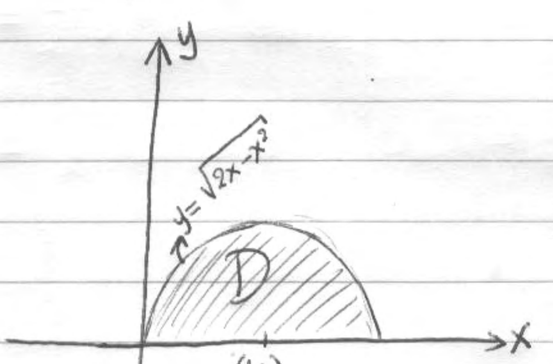
• $I=\iint_D (x^2+y^2)\,dxdy$

D ημικύκλιο $\quad x^2+y^2=2x$, $y\geq 0$

$\rightarrow$ (εξ. περιφερ. $\quad x^2+y^2+Ax+By+C=0$

$K=(h,k)\quad h=-\frac{A}{2}\quad k=-\frac{B}{2}$

$r=\sqrt{h^2+k^2-c}$)



$\left.\begin{array}{l} x = r\sigma\upsilon\nu\vartheta \\ y = r\eta\mu\vartheta \end{array}\right\}$ $x^2+y^2=2x \Rightarrow$

$r^2 = 2r\sigma\upsilon\nu\vartheta \Rightarrow$

$r = 2\sigma\upsilon\nu\vartheta$

$D=\{(r,\vartheta): 0\le\vartheta\le\pi/2,\ 0\le r\le 2\text{συν}\vartheta\}$ ϑ-απλό

$$I=\int_0^{\pi/2}\int_0^{2\text{συν}\vartheta} r^2\, r\, dr\, d\vartheta=\int_0^{\pi/2}\int_0^{2\text{συν}\vartheta} r^3\, dr\, d\vartheta=$$

$$=\int_0^{\pi/2}\frac{r^4}{4}\Big|_0^{2\text{συν}\vartheta}d\vartheta=2\int_0^{\pi/2}\text{συν}^4\vartheta\, d\vartheta=$$

$$=2\int_0^{\pi/2}(\text{συν}^2\vartheta)^2d\vartheta=2\int_0^{\pi/2}\left(\frac{1+\text{συν}2\vartheta}{2}\right)d\vartheta$$

(ή $\int\text{συν}^4\vartheta\, d\vartheta=\int\text{συν}^3\vartheta\,\text{συν}\vartheta\, d\vartheta=\int\text{συν}^3\vartheta\,(\text{ημ}\vartheta)'\, d\vartheta\ldots$)

$m=\iint_D \delta(x,y)\,dx\,dy$ μάζα

$M_x=\iint_D y\,\delta(x,y)\,dx\,dy$

$M_y=\iint_D x\,\delta(x,y)\,dx\,dy$

στατικές ροπές

$\bar{x}=\frac{M_y}{m}$ $\bar{y}=\frac{M_x}{m}$

$K(\bar{x},\bar{y})$ (κέντρο μάζας)

$I_x=\iint_D y^2\delta(x,y)\,dx\,dy$

$I_y=\iint_D x^2\delta(x,y)\,dx\,dy$

$I_0=I_x+I_y$

ροπές αδράνειας

Παράδειγμα 10.25